

On D. Hägele's approach to the Bessis-Moussa-Villani conjecture

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Abstract. The reformulation of the Bessis-Moussa-Villani conjecture given by Lieb and Seiringer asserts that the coefficient $\alpha_{p,r}(A, B)$ of t^r in the polynomial $\text{Tr}(A + tB)^p$, with A, B positive semidefinite matrices, is nonnegative for all p, r . We propose a natural extension of a method of attack on this problem due to Hägele, and investigate for what values of p, r the method is successful, obtaining a complete determination when either p or r is odd.

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1. Introduction

In [3], Daniel Hägele gives an ingenious and simple proof that if A and B are $n \times n$ positive semidefinite matrices then for $p = 7$ all coefficients $\alpha_{p,r}(A, B)$ of t in the polynomial

$$\text{Tr}(A + tB)^p \equiv \sum_{r=0}^p \alpha_{p,r}(A, B)t^r, \quad (1)$$

where $\text{Tr } M$ denotes the trace of the matrix M , are nonnegative. If this result could be proved for general p it would imply [7] a conjecture of Bessis, Moussa, and Villani [1]. On the other hand, it was also shown in [3] that the same method does not suffice to prove the positivity of $\alpha_{6,3}$ (we will occasionally abbreviate “ $\alpha_{p,r}(A, B) \geq 0$ for all positive semidefinite A, B ” as “ $\alpha_{p,r} \geq 0$ ”). Thus it is of interest to investigate for what values of p and r the method does or does not succeed in establishing $\alpha_{p,r} \geq 0$.

In this note we give several results, both negative and positive, in this direction. We must to some extent consider separately two possible cases, according to the parity of p and r , and in each of these cases we define two related integers k and q :

Case 1: p and r are odd. Then $p = 2k + 1$, $r = 2q + 1$;

Case 2: p is even and r is odd. Then $p = 2k + 2$, $r = 2q + 1$.

One further case,

Case 3: p is odd and r even,

is included implicitly; it is easy to verify that all our results for Case 1 imply corresponding results for Case 3, obtained by replacing r with $p - r$. We will not consider in detail the

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case in which both p and r are even; results in this case have been obtained by Klep and Schweighofer [5,6] and by Burgdorf [2], as we discuss briefly in Section 4. In each of Cases 1 and 2 we define precisely a proof strategy which is the natural generalization of that of [3] and investigate its success. We are able to classify completely the pairs (p, r) for which the method succeeds; unfortunately, although these include one infinite class $(p, r \text{ odd with } r = p - 4)$, the method does not succeed in enough cases to establish the BMV conjecture.

Results of this sort should be viewed in the light of an important theorem of Hillar [4], which implies that if $\alpha_{p,r} \geq 0$ then also $\alpha_{p',r'} \geq 0$ if $p \geq p'$, $r \geq r'$, and $p - r \geq p' - r'$. For example, it is pointed out in [3] that although the proof method used there does not apply directly when $p = 6$, $r = 3$, the nonnegativity of $\alpha_{6,3}$ follows from the corresponding result for $p = 7$, $r = 3$; similarly, our result that $\alpha_{p,p-4} \geq 0$ for p odd implies the positivity of $\alpha_{p,r}$, for all p , when $r \leq 4$ or $r \geq p - 4$. Moreover, it follows that to establish the full BMV conjecture it suffices to establish positivity of α_{p_n,r_n} for some sequences p_n, r_n with $p_n \rightarrow \infty$, $r_n \rightarrow \infty$, and $p_n - r_n \rightarrow \infty$ as $n \rightarrow \infty$. Our results leave open the possibility of proving the BMV conjecture by successfully applying the method of [3] to such a sequence with p_n, r_n even.

In order to describe the method more precisely we write $X_0 \equiv A$ and $X_1 \equiv B$. Let $E_{p,r}$ be the set of binary strings of length p , $s = s_1 \cdots s_p$, containing exactly r 1's, and for $s \in E_{p,r}$ write $Y_s = X_{s_1} \cdots X_{s_p}$. Then

$$\alpha_{p,r}(A, B) = \sum_{s \in E_{p,r}} \text{Tr}(Y_s). \quad (2)$$

Now for coefficients $c = (c_u)_{u \in E_{k,q}} \in \mathbb{C}^{E_{k,q}}$ define $Z(c) = \sum_{u \in E_{k,q}} c_u Y_u$. Then we will have $\alpha_{p,r}(A, B) \geq 0$ if we can show that for some appropriately chosen $c^{(m)} = (c_u^{(m)})_{u \in E_{k,q}}$, $1 \leq m \leq M$,

$$\alpha_{p,r}(A, B) = \begin{cases} \sum_m \text{Tr}(Z(c^{(m)})BZ(c^{(m)})^*), & \text{in case 1,} \\ \sum_m \text{Tr}(Z(c^{(m)})BZ(c^{(m)})^*A), & \text{in case 2.} \end{cases} \quad (3)$$

This follows from the fact that if a and b are the nonnegative square roots of A and B , respectively, then $\text{Tr}(Z(c)BZ(c)^*) = \text{Tr}[(Z(c)b)(Z(c)b)^*]$ and $\text{Tr}(Z(c)BZ(c)^*A) = \text{Tr}[(aZ(c)b)(aZ(c)b)^*]$.

To relate (2) with (3) we must make explicit the effect of the invariance of the trace under cyclic permutations. Let $\tilde{E}_{p,r}$ be the set of equivalence classes of $E_{p,r}$ modulo cyclic permutations, with $\pi : E_{p,r} \rightarrow \tilde{E}_{p,r}$ the canonical projection. Then (2) becomes

$$\alpha_{p,r}(A, B) = \sum_{\tilde{s} \in \tilde{E}_{p,r}} |\tilde{s}| \text{Tr}(Y_{s(\tilde{s})}), \quad (4)$$

where $|\tilde{s}|$ is the number of elements in \tilde{s} and $s(\tilde{s})$ is some element of \tilde{s} . Similarly, if we define $\sigma \equiv \sigma_{p,r} : E_{k,q} \rightarrow E_{p,r}$ by

$$\sigma_{p,r}(u, v) = \begin{cases} u_1 \cdots u_k 1 v_k \cdots v_1, & \text{in case 1,} \\ u_1 \cdots u_k 1 v_k \cdots v_1 0, & \text{in case 2,} \end{cases}$$

then the right hand side of (3) becomes

$$\sum_{\tilde{s} \in \tilde{E}_{p,r}} \sum_{(u,v) \in (\pi\sigma)^{-1}(\tilde{s})} \sum_m c_u^{(m)} \bar{c}_v^{(m)} \text{Tr}(Y_{\sigma(u,v)}), \quad (5)$$

so that (3) will hold for all A, B if for all $\tilde{s} \in \tilde{E}_{p,r}$,

$$|\tilde{s}| = \sum_{(u,v) \in (\pi\sigma)^{-1}(\tilde{s})} \sum_m c_u^{(m)} \bar{c}_v^{(m)}. \quad (6)$$

The generalization of the method of [3] referred to above is establish

Condition H: There exist $M \geq 1$ and coefficients $c^{(m)}$, $m = 1, \dots, M$, such that (6) is satisfied for all $\tilde{s} \in \tilde{E}_{p,r}$.

Before proceeding we verify a fact which is obviously necessary for the existence of such $c^{(m)}$.

Proposition 1: For any p and r and any $\tilde{s} \in \tilde{E}_{p,r}$ there exist $u, v \in \tilde{E}_{k,q}$ such that $\sigma_{p,r}(u, v) \in \tilde{s}$.

Proof: We give the proof in Case 1; Case 2 is similar. A useful geometric picture (the reader might draw a sketch) is obtained by letting $C \subset \mathbb{C}$ denote the set of p^{th} roots of unity and identifying an element $s = s_1 \cdots s_p \in E_{p,r}$ with a map $s : C \rightarrow \{0, 1\}$ labeling the elements of C ; the identification is via $s(\exp 2j\pi i/p) = s_j$, $j = 1, \dots, p$. Any $\theta \in \mathbb{R}$ defines the line L_θ in \mathbb{C} through the origin and the point $z_\theta = \exp i\theta$, oriented from the origin toward z_θ . Let $N_1(\theta)$ be the number of points $\omega \in C$ for which $s(\omega) = 1$ and which lie to the right of L_θ , let $N_2(\theta)$ be the number of such points which lie to the left of L_θ , and let $N(\theta) = N_1(\theta) - N_2(\theta)$. $N(\theta)$ is odd unless $\pm z_\theta \in C$ with $s(\pm z_\theta) = 1$, in which case it is even, and if $N(\theta_0) = 0$ for some θ_0 then we can immediately read off the desired u, v . But taking θ with $\pm z_\theta \notin C$, so that $N(\theta)$ is odd, we observe that $N(\theta + \pi) = -N(\theta)$ and so $N(\theta_0) = 0$ for some intermediate θ_0 . ■

2. Positive results

In this section we show that Condition H holds in the following cases:

Case 1: $r = 1$; $r = p - 2$; $r = p - 4$; and $p = 11, r = 3$. The cases $r = 1$ and $r = p - 2$ are easy (in each case one takes $M = 1$ and $c_u^{(1)} = 1$ for all $u \in E_{k,q}$); the remaining cases are covered in Theorems 2 and 3 below.

Case 2: $r = 1$ and $r = p - 1$. These follow the pattern of the two easy cases above; verification is left to the reader.

Theorem 2: Condition H holds if $p = 11$ and $r = 3$.

Proof: Defining

$$\begin{aligned}
Z_1 &= Y_{00001} + Y_{00010} + Y_{00100} + Y_{01000} - Y_{10000}, \\
Z_2 &= \sqrt{2}(Y_{00100} - Y_{01000} - Y_{10000}), \\
Z_3 &= 2(Y_{00100} - Y_{01000}), \\
Z_4 &= 2Y_{01000},
\end{aligned} \tag{7}$$

and using the fact that, since $p = 11$ is prime, $|\tilde{s}| = 11$ for all $\tilde{s} \in \tilde{E}_{11,3}$, one finds easily that (compare (3), case 1)

$$\alpha_{11,3}(A, B) = 11 \sum_{i=1}^4 \text{Tr}(Z_i B Z_i^*). \quad \blacksquare$$

We remark that both positive and negative coefficients occur among the $c_u^{(m)}$ implicitly defined by (7). It can easily be shown that no solution in which all the coefficients are positive is possible; this is in contrast to the situation for the case $p = 7$, $r = 3$ discussed in [3] and for the cases treated in Theorem 3 below.

Theorem 3: *Condition H holds if p is odd and $r = p - 4$.*

Note that the case $p = 7$, $r = 3$ of this theorem appears in [3]; the case $p = 9$, $r = 5$ was obtained by Klep and Schweighofer (see [5]). After we had completed our work we learned that Theorem 3 was obtained independently by Burgdorf [2].

The theorem will follow almost immediately from the next lemma.

Lemma 4: *Let $p = 2k + 1 \geq 5$ and let $r = p - 4 = 2q + 1$. Then $E_{k,q}$ may be partitioned as $E_{k,q} = \bigcup_{m=1}^{k-1} D_m$ in such a way that for every $\tilde{s} \in \tilde{E}_{p,r}$ there exists a unique m , $1 \leq m \leq k - 1$, and unique $u, v \in D_m$, such that $\sigma(u, v) \in \tilde{s}$.*

Proof of Theorem 3: Set $p = 2k + 1$ and $p - 4 = 2q + 1$. We must find coefficients $c^{(m)} = (c_u^{(m)})_{u \in E_{k,q}}$ satisfying (6); since p and $p - 4$ are relatively prime, $|\tilde{s}| = p$ for every $\tilde{s} \in \tilde{E}_{p,r}$ and so equivalently we must find $c^{(m)}$ satisfying

$$\sum_m \sum_{(u,v) \in (\pi\sigma)^{-1}(\tilde{s})} c_u^{(m)} \bar{c}_v^{(m)} = 1. \tag{8}$$

But from Lemma 4, (8) holds if $c^{(m)}$, $m = 1, \dots, k - 1$, is the characteristic function of D_m : $c_u^{(m)} = 1$ if $u \in D_m$, $c_u^{(m)} = 0$ otherwise. \blacksquare

The next proof is somewhat complicated; it might help the reader to work through it in the case $p = 9$, $r = 5$ (this was the case that suggested the general result).

Proof of Lemma 4: Recalling that an element $u \in E_{k,q}$ is a binary string $u_1 u_2 \cdots u_k$, we define

$$\begin{aligned}
D_1 &= \{u \in E_{k,q} \mid u_1 = 0\}, \\
D_2 &= \{u \in E_{k,q} \mid u_1 = 1, u_k = 0\}, \\
D_3 &= \{u \in E_{k,q} \mid u_1 = u_k = 1, u_2 = 0\}, \\
D_4 &= \{u \in E_{k,q} \mid u_1 = u_k = u_2 = 1, u_{k-1} = 0\}, \quad \text{etc.},
\end{aligned}$$

and in general, for $j \geq 0$,

$$\begin{aligned} D_{2j+1} &= \{u \in E_{k,q} \mid u_1 = u_2 = \cdots = u_j = u_k = u_{k-1} = \cdots = u_{k-j+1} = 1, u_{j+1} = 0\}, \\ D_{2j+2} &= \{u \in E_{k,q} \mid u_1 = u_2 = \cdots = u_{j+1} = u_k = u_{k-1} = \cdots = u_{k-j+1} = 1, u_{k-j} = 0\}. \end{aligned}$$

It is clear that the D_m so defined form a partition of $E_{k,q}$. We will write $\tilde{D}_m = \sigma(D_m \times D_m)$, so that we must prove that for any $\tilde{s} \in \tilde{E}_{p,r}$, $|\tilde{s} \cap \bigcup_{m=1}^{k-1} \tilde{D}_m| = 1$.

Note that a string $u \in E_{k,q}$ contains exactly two zeros, and if $u \in D_m$ then the position of one of these zeros is fixed and there are $k - m$ possible positions for the remaining one; thus $|D_m| = k - m$. Note also that $u, v \in D_m$ if and only if the form of $\sigma(u, v)$ is

$$1^j 0 w 1^j 1 1^j x 0 1^j, \quad \text{if } m = 2j + 1, j \geq 0, \quad (9a)$$

$$1^{j+1} w 0 1^j 1 1^j 0 x 1^{j+1}, \quad \text{if } m = 2j + 2, j \geq 0 \quad (9b)$$

where $w, x \in E_{k-m, k-m-1}$ are arbitrary.

Now fix $\tilde{s} \in \tilde{E}_{p,r}$. There are nonnegative integers n_0, \dots, n_3 , with $n_0 + n_1 + n_2 + n_3 = 2k - 3$, such that \tilde{s} consists of all cyclic permutations of the string

$$0 1^{n_0} 0 1^{n_1} 0 1^{n_2} 0 1^{n_3}. \quad (10)$$

We must show that precisely one element of \tilde{s} has one of the forms (9).

Consider first (9a); the initial $1^j 0$ and final $0 1^j$ there imply that if that string is put in the form (10) by a cyclic permutation then it will contain a substring $0 1^{2j} 0$, i.e., that if an element in \tilde{s} has the form (9a) then one of the integers n_i must be even. Conversely, if n_i is even for some i , with $n_i = 2j_i$ ($j_i \geq 0$), then the string $s_i \in \tilde{s}$ defined by

$$s_i = 1^{j_i} 0 1^{n_{i+1}} 0 1^{n_{i+2}} 0 1^{n_{i+3}} 0 1^{j_i} \quad (11a)$$

(here addition on the indices of the n_l 's is taken modulo 4) will lie in $\tilde{D}_{n_{i+1}}$ if n_{i+1} and n_{i-1} satisfy certain additional constraints, which we discuss below. The discussion of (9b) is similar: if some n_i is odd, $n_i = 2j_i + 1$ ($j_i \geq 0$), then the cyclic permutation of (10) in which the block 1^{n_i} is moved to the center is a candidate to lie in $\tilde{D}_{n_{i+1}}$. If $j_i + n_{i-1} + 2 \leq k$ and $j_i + n_{i+1} + 2 \leq k$ (the only case that will be relevant, since (9b) has two zeros on each side of its center) then this string has the form

$$s_i = 1^{k-(j_i+n_{i-1}+2)} 0 1^{n_{i-1}} 0 1^{j_i} 1 1^{j_i} 0 1^{n_{i+1}} 0 1^{k-(j_i+n_{i+1}+2)}, \quad (11b)$$

and will lie in $\tilde{D}_{n_{i+1}}$ under further constraints on $n_{i \pm 1}$. We see that for each i , $i = 0, 1, 2, 3$, there is one possible element of \tilde{s} which could lie in $\tilde{D}_{n_{i+1}}$, given by (11a) or (11b) as n_i is even or odd.

Now we ask what further conditions on $n_{i \pm 1}$ would imply that (11a) has the form (9a) or (11b) the form (9b). Consider first (11a), and recall that here $n_i = 2j_i$. The second zero in (11a) is located at position $j_i + n_{i+1} + 2$, and for (11a) to have the form (9a) it is necessary that this zero lie to the left of a block $1^{j_i} 1 1^{j_i}$ at the center of the string, that is, to the left of position $k - j_i + 1$. Thus $s_i \in \tilde{D}_{n_{i+1}}$ is possible only if $j_i + n_{i+1} + 2 < k - j_i + 1$,

i.e., only if $n_i + n_{i+1} \leq k - 2$. Combining this result with that of a similar analysis of the position of the third zero shows that

$$s_i \in \tilde{D}_{n_i+1} \text{ if and only if } n_i + n_{i+1} \leq k - 2 \text{ and } n_i + n_{i-1} \leq k - 2. \quad (12)$$

The analysis of (11b), where $n_i = 2j_i + 1$, is similar: for this to have the form (9b), there must be at least $j_i + 1$ initial ones in the string, requiring that $k - (j_i + n_{i-1} + 2) \geq j_i + 1$; since there must also be $j_i + 1$ ones at the end of the string we are led again to the conclusion (12).

Finally we observe that the condition that $\sum_{i=0}^3 n_i = 2k - 3$ implies that of any pair of inequalities $n_i + n_{i+1} \leq k - 2$ and $n_{i+2} + n_{i+3} \leq k - 2$ exactly one must be true. This implies that the condition of (12) will be satisfied for exactly one value of i (modulo 4), so that $s_i \in \tilde{D}_{n_i+1}$ (that is, $\tilde{s} \cap \tilde{D}_{n_i+1} = \{s_i\}$) holds for precisely one value of i . From (11a) or (11b) one can then read off the unique $u, v \in D_{n_i+1}$ such that $\sigma(u, v) = s_i$. ■

3. Negative results

In this section we show that Condition H does not hold in the following cases:

Case 1: $5 \leq r \leq p - 6$; $p \geq 13$, $r = 3$; and $p = 9$, $r = 3$.

Case 2: $3 \leq r \leq p - 3$.

The method of proof in all of these cases is similar to the argument of [3] establishing a negative result for $p = 6$, $r = 3$.

Throughout the rest of this section we assume that we are in case 1 or case 2, that is, that $r = 2q + 1$ is odd, but to the extent possible we treat these two cases in a unified manner, so that for the moment either $p = 2k + 1$ or $p = 2k + 2$. If $u, v \in E_{k,q}$ we write $\tilde{N}(u, v) = |\pi(\sigma(u, v))|$ and $N(u, v) = |(\pi\sigma)^{-1}(\pi(\sigma(u, v)))|$; that is, $\tilde{N}(u, v)$ is the number of distinct strings obtained from $\sigma(u, v)$ by cyclic permutation, and $N(u, v)$ is the number of ordered pairs $(w, x) \in E_{k,q} \times E_{k,q}$ such that $\sigma(w, x)$ is obtained from $\sigma(u, v)$ by a cyclic permutation. We will compute $N(u, v)$ using the following simple remark.

Remark 5: Let $k' = p - k - 1$ so that $k' = k$ in case 1, $k' = k + 1$ in case 2. Then for any $s \in E_{p,r}$ with $|\pi(s)| = p$, $|(\pi\sigma)^{-1}(\pi(s))|$ is equal to the number of indices i , $1 \leq i \leq p$, such that (i) $s_i = 1$ and the preceding (if $i \geq k' + 1$) or succeeding (if $i \leq p - k'$) k' entries of s —that is $s_{i-k'} \cdots s_{i-1}$ or $s_{i+1} \cdots s_{i+k'}$, respectively—contain exactly q ones, and (ii) in case 2, if also $i - k' = 0$ or $i + k' = 0$, respectively. Of course if $s = \sigma(u, v)$ then $i = k + 1$ satisfies this criterion. The application of this remark in any particular case is straightforward but tedious; we give a full discussion of one case in the proof of Lemma 6 and after that we are rather sketchy, leaving the details to the reader. It is probably most helpful to work out a simple example in each case.

We now define $w = 0^{k-q} 1^q \in E_{k,q}$.

Lemma 6: Suppose that $u \in E_{k,q}$. Then (a) $\tilde{N}(w, u) = p$, and (b) if $u_1 = 0$ or p is even (i.e., we are in case 2) then $N(w, u) = 1$. In particular, (c) $\tilde{N}(w, w) = p$ and $N(w, w) = 1$.

Proof: (a) The string $\sigma(w, u)$ contains a substring of at least $q + 1$ consecutive ones, and since there are a total of $2q + 1$ ones in the string, no nontrivial cyclic permutation of $\sigma(w, u)$ can coincide with it.

(b) Under either hypothesis, $s \equiv \sigma(w, u)$ has the form $s = 0^{k-q} 1^q 1 s_{k+2} \cdots s_{p-1} 0$; the key observation is that for $1 \leq j \leq q + 1$ the last j entries of s can contain at most $j - 1$ ones, and so entries $k + 2, \dots, p - j$ must contain at least $q - j + 1$ ones. We show that no index i , $1 \leq i \leq p$, other than $i = k + 1$, can satisfy criterion (i) of Remark 5. Suppose then that $s_i = 1$ and $i \neq k + 1$. There are three possible cases: if $k - q + 1 \leq i \leq k$ then $s_{i+1} \cdots s_{i+k'} = 1^{k-i} 1 s_{k+2} \cdots s_{p-(k+1-i)}$ contains, by the observation above, at least $(k - i) + 1 + (q - k + i) = q + 1$ ones; if $k + 2 \leq i \leq p - q - 1$ then $s_{i-k'} \cdots s_{i-1}$ contains the substring $s_{k-q+1} \cdots s_{k+1} = 1^{q+1}$; and if $p - q \leq i \leq p$ then $s_{i-k'} \cdots s_{i-1} = 1^{p-i} 1 s_{k+2} \cdots s_{i-1}$ contains at least $(p - i) + 1 + (q - p + i) = q + 1$ ones.

(c) This is an immediate consequence of (a) and (b). ■

Lemma 7: Suppose there exist $x, y, z \in E_{k,q}$, all distinct from w and with $x \neq y$ and $x \neq z$, such that

$$N(w, x) = N(w, y) = N(x, x) = 1, \quad (13a)$$

$$\tilde{N}(w, x) = \tilde{N}(w, y) = \tilde{N}(x, x) = p, \quad (13b)$$

$$N(z, z) = 3, \text{ with } \pi(\sigma(z, z)) = \{\sigma(z, z), \sigma(x, y), \sigma(y, x)\}. \quad (13c)$$

Then Condition H does not hold.

We remark that the requirement that all of x, y, z and w be distinct, except for the possibility that $y = z$, actually follows from (13) and Lemma 6.

Proof: We suppose that for some M and $c^{(m)}$, (6) holds for all \tilde{s} , and derive a contradiction. From (6) applied to $\pi(\sigma(w, w))$, $\pi(\sigma(x, x))$, and $\pi(\sigma(w, x))$ we have, using Lemma 6(c) and (13a)–(13b),

$$p = \sum_m c_w^{(m)} \bar{c}_w^{(m)} = \sum_m c_x^{(m)} \bar{c}_x^{(m)} = \sum_m c_w^{(m)} \bar{c}_x^{(m)}. \quad (14)$$

These equations, together with the standard necessary condition for equality to hold in the Cauchy-Schwarz inequality, then imply that

$$c_w^{(m)} = c_x^{(m)}, \quad \text{for } m = 1, \dots, M. \quad (15)$$

But, first from (6) applied to $\pi(\sigma(w, y))$, and then from (15),

$$p = \sum_m c_w^{(m)} \bar{c}_y^{(m)} = \sum_m c_x^{(m)} \bar{c}_y^{(m)}. \quad (16)$$

Finally, from (6) applied to $\pi(\sigma(z, z))$, (13c), and then (16),

$$\tilde{N}(z, z) = \sum_m c_z^{(m)} \bar{c}_z^{(m)} + \sum_m c_x^{(m)} \bar{c}_y^{(m)} + \sum_m c_y^{(m)} \bar{c}_x^{(m)} = \sum_m c_z^{(m)} \bar{c}_z^{(m)} + 2p \geq 2p, \quad (17)$$

a contradiction, since $\tilde{N}(z, z)$ must divide p . ■

Theorem 8: *If r is odd and (a) p is odd and $5 \leq r \leq p - 6$, (b) p is odd, $p \geq 13$, and $r = 3$, or (c) p is even and $3 \leq r \leq p - 3$, then Condition H does not hold.*

Proof: (a) In this case we claim that the strings

$$x = 010^{k-q-1}1^{q-1}, \quad y = 0^{k-q-2}1^q0^2, \quad \text{and} \quad z = 01^q0^{k-q-1},$$

fulfill the conditions of Lemma 7. Since $2 \leq q \leq k - 3$ we have $x \neq y$ and $x \neq z$ (although $y = z$ if $q = k - 3$). The conditions

$$N(w, x) = N(w, y) = 1, \quad \tilde{N}(w, x) = \tilde{N}(w, y) = p,$$

follow from Lemma 6, since $x_1 = y_1 = 0$.

Consider now $\sigma(x, x) = 010^{k-q-1}1^{q-1}11^{q-1}0^{k-q-1}10$; this contains a unique string of $2q - 1 \geq 3$ consecutive ones and so can never coincide with a cyclic permutation of itself, so that indeed $\tilde{N}(x, x) = p$. A detailed analysis using Remark 5, as in the proof of Lemma 6(b) (but by symmetry it is necessary to consider only $i \leq k$), shows that $N(x, x) = 1$.

Finally consider $s \equiv \sigma(z, z) = 01^q0^{k-q-1}10^{k-q-1}1^q0$. Again, consideration of the sizes of the three blocks of consecutive ones shows that $\tilde{N}(z, z) = p$. To find $N(z, z)$ we note that a cyclic permutation which brings the one at position $i = 2$ of s to the center position $i = k + 1$ yields that string $0^{k-q-2}1^q0^211^{q-1}0^{k-q-1}10 = \sigma(y, x)$, and one obtains $\sigma(x, y)$ by a cyclic permutation bringing the one at $i = p - 1$ in s to $i = k + 1$. However, if $3 \leq i \leq q + 1$ then $s_{i+1} \cdots s_{i+k}$ contains at most $q - 1$ ones, with a similar conclusion if $p - q - 1 \leq i \leq p - 2$, so that $N(z, z) = 3$ and (13c) holds.

(b) In this case the strings

$$x = 0^{k-3}10^2, \quad y = z = 010^{k-2},$$

fulfill the conditions of Lemma 7; the verification is similar to the above.

(c) If $5 \leq r \leq p - 3$ then the strings

$$x = 10^{k-q}1^{q-1}, \quad y = 0^{k-q-1}1^q0, \quad z = 1^q0^{k-q},$$

fulfill the conditions of Lemma 7; again the verification is similar to that of case (a). If $r = 3$ and $p \geq 8$ then the conclusion follows from the case $r = p - 3$ after the interchange of A and B . Finally, the result for case $p = 6$, $r = 3$ was established in [3]. ■

The next result covers the one remaining negative result not included in Theorem 8. It is stated without proof in [5].

Theorem 9: *If $p = 9$ and $r = 3$ then Condition H does not hold.*

Proof: Again we suppose that there exist $c^{(m)}$, $m = 1, \dots, M$, so that (6) holds for all \tilde{s} , and derive a contradiction by looking at a few specific choices of \tilde{s} , as given in Table 1; there we write $v_1 = 0001$, $v_2 = 0100$ (with $v_1, v_2 \in E_{4,1}$).

Name of \tilde{s}	Typical $s \in \tilde{s}$	$ \tilde{s} $	$(\pi\sigma)^{-1}(\tilde{s})$
\tilde{s}_1	000111000	9	$\{(v_1, v_1)\}$
\tilde{s}_2	010010010	3	$\{(v_2, v_2)\}$
\tilde{s}_3	000110010	9	$\{(v_1, v_2)\}$

Table 1

From (6) applied to \tilde{s}_1 , \tilde{s}_2 , and \tilde{s}_3 , we have

$$9 = \sum_m c_{v_1}^{(m)} \bar{c}_{v_1}^{(m)} = \sum_m c_{v_1}^{(m)} \bar{c}_{v_2}^{(m)}; \quad 3 = \sum_m c_{v_2}^{(m)} \bar{c}_{v_2}^{(m)}. \quad (18)$$

These equations, however, are inconsistent with the Cauchy-Schwarz inequality. ■

4. Concluding remarks

In recent work [6] Klep and Schweighofer give a systematic algebraic language in which to discuss the method of [3]. They introduce the associative \mathbb{R} -algebra $\mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle$ with noncommuting generators \mathbf{a} and \mathbf{b} (X and Y in the notation of [6]), furnished with a natural involution $f \mapsto f^*$ obtained by reversing each word in the generators. They further define $\Sigma^2 \subset \mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle$ to be the cone of elements $f \in \mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle$ which may be written as sums of Hermitian squares, $f = \sum_i g_i^* g_i$, and Θ^2 to be the cone of elements which are cyclically equivalent to elements of Σ^2 , where two elements f and g are cyclically equivalent if their difference is a sum of commutators. It follows that if $f(\mathbf{a}, \mathbf{b}) \in \Theta^2$ and a, b are nonnegative $n \times n$ matrices then $\text{Tr}(f(a, b)) \geq 0$, so in order to show that $\alpha_{p,r} \geq 0$ it suffices to verify that $S_{p,r}(\mathbf{a}^2, \mathbf{b}^2) \in \Theta^2$, where $S_{p,r}(\mathbf{a}^2, \mathbf{b}^2) \in \mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the sum of all possible products of r factors \mathbf{b}^2 and $p - r$ factors \mathbf{a}^2 .

It is immediate that if hypothesis H is satisfied for some p, r falling under Case 1 or Case 2, or if H is satisfied for $p, p - r$ with p, r falling under Case 3, then $S_{p,r}(\mathbf{a}^2, \mathbf{b}^2) \in \Theta^2$; further, it follows from a result of [6] (Proposition 2.2) that the converse also holds. This means that the results of Sections 2 and 3 establish, for every p, r with either p or r odd, whether or not $S_{p,r}(\mathbf{a}^2, \mathbf{b}^2) \in \Theta^2$. In particular, we can conclude that the approach of [3] (at least as formulated in [6]) when applied to such p and r cannot establish the BMV conjecture for any p larger than 9.

Thus to make progress on the BMV conjecture using this approach one must consider cases in which both p and r are even. In this direction, Klep and Schweighofer show [6] that $S_{14,4}(\mathbf{a}^2, \mathbf{b}^2)$ and $S_{14,6}(\mathbf{a}^2, \mathbf{b}^2)$ belong to Θ^2 , which, together with results of [4] or by independent arguments given in [6], implies that the BMV conjecture is satisfied for $p = 13$ and indeed, by [4], for $p \leq 13$. Moreover, Burgdorf [2] has obtained a version of Theorem 3 strengthened to include p, r even: she shows that $S_{p,4}(\mathbf{a}^2, \mathbf{b}^2) \in \Theta^2$ (and hence $S_{p,p-4}(\mathbf{a}^2, \mathbf{b}^2) \in \Theta^2$) for all $p \geq 4$.

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